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INVERSE SCATTERING WITH FIXED ENERGY AND AN INVERSE EIGENVALUE PROBLEM ON THE HALF-LINE

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Abstract. Recently A. G. Ramm (1999) has shown that a subset of phase shifts δ_l, $l = 0, 1, \dots$, determines the potential if the indices of the known shifts satisfy the Müntz condition $\sum_{l\neq 0, l\in L} \frac{1}{l} = \infty$. We prove the necessity of this condition in some classes of potentials. The problem is reduced to an inverse eigenvalue problem for the half-line Schrödinger operators.

1. Introduction

Consider the inverse scattering problem for the operator

$$\varphi''(r) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r) - q(r)\varphi(r) + k^2\varphi(r) = 0, \quad r \geq 0, \tag{1.1}$$

with fixed energy $k^2 = 1$. Suppose that the potential q satisfies $rq(r) \in L_1(0, \infty)$. It is known [6] that for $\Re\lambda \geq 0$ there exists a solution of (1.1), unique up to a constant multiple, satisfying the boundary conditions

$$\varphi(r, \lambda) = \mathbf{O}(r^{\lambda+1/2}), \qquad r \to 0+, \tag{1.2}$$

$$\varphi(r, \lambda) = A(\lambda)\sin(r - \pi/2 \cdot (\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1), \qquad r \to \infty. \tag{1.3}$$

The values $\delta(\lambda) \in \mathbf{C}$ are called phase shifts; they are defined by (1.3) only mod π. In quantum mechanics most relevant are the shifts

$$\delta_l = \delta(l + 1/2), \qquad l = 0, 1, 2, \dots. \tag{1.4}$$

Concerning the recovery of the potential $q(r)$ by a set of phase shifts $\delta(\lambda_n)$ we mention the following recent result of Ramm.

Theorem 1.1 ([9]). *Suppose that $q(r) = 0$ for $r > a$ and $rq(r) \in L_2(0, a)$. Let $L \subset \mathbf{N}$ and suppose that the Müntz condition*

$$\sum_{l\neq 0, l\in L} \frac{1}{l} = \infty \tag{1.5}$$

is valid. Then the data δ_l, $l \in L$, uniquely determine the potential $q(r)$.

We show next that here L_2 can be substituted by L_1 and that the condition (1.5) is "almost" necessary.

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Theorem 1.2. *Let $q(r) = 0$ for $r > a$ and $rq(r) \in L_1(0,a)$. Consider arbitrary different complex numbers λ_n with $\Re\lambda_n \geq \gamma > 0$. If*

$$\sum \frac{\Re\lambda_n}{|\lambda_n|^2} = \infty, \tag{1.6}$$

then the potential $q(r)$ is uniquely determined by the shifts $\delta(\lambda_n)$.

Theorem 1.3. *Let $0 < \sigma < 2$ be arbitrarily fixed and consider the class*

$$B_\sigma = \{q(r) : q(r) = 0 \text{ for } r > a,\ r^{1-\sigma}q(r) \in L_1(0,a)\}.$$

Finally let $\lambda_n \in \mathbf{C}$ be arbitrary numbers with $\Re\lambda_n \geq \gamma > 0$. Now if

$$\sum \frac{\Re\lambda_n}{|\lambda_n|^2} < \infty, \tag{1.7}$$

then the shifts $\delta(\lambda_n)$ are not enough to recover the potential in B_σ; in other words, for every $q \in B_\sigma$ there exists a different $q^* \in B_\sigma$ such that $\delta(q^*,\lambda_n) = \delta(q,\lambda_n)$ $\forall n$.

Remark. If the numbers λ_n are of the form $l + 1/2$, then (1.5) and (1.6) are equivalent. Thus Theorem 1.2 extends Theorem 1.1.

Remark. The potentials of Theorem 1.1 belong to B_σ for every $0 < \sigma < 1/2$. Hence if (1.5) does not hold, then there exists another potential $q^* \in B_\sigma$ with the same data δ_l, $l \in L$, as for q. Whether there exists a potential q^* with $rq^*(r) \in L_2(0,a)$ and $\delta_l(q^*) = \delta_l(q)$, $l \in L$, is an open question. Analogous L_p-problems are not investigated here.

Next we consider the inverse eigenvalue problem for the Schrödinger operators

$$-y'' + Q(x)y = \lambda^2 y, \quad x \geq 0. \tag{1.8}$$

It is known that for $Q \in L_1(0,\infty)$ the operator is a limit point at infinity and the (essentially unique) L_2-solution satisfies the asymptotical formula

$$y(x,\lambda) = c(\lambda)e^{i\lambda x}(1 + \mathbf{o}(1)) \quad x \to \infty, \quad \Im\lambda > 0; \tag{1.9}$$

see Theorem 2.1 below. Consider the boundary condition

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0 \tag{1.10}$$

for some $0 \leq \alpha < \pi$. The values λ^2 for which the system (1.8), (1.10) has a nontrivial L_2-solution are called eigenvalues of the Schrödinger operator (1.8) with boundary condition (1.10). It is known that the eigenvalues are negative. We apply the notation

$$\lambda^2 \in \sigma(Q,\alpha)$$

for the eigenvalues of (1.8), (1.10).

We say that the values $\lambda_n^2 < 0$ are *common eigenvalues* of the potentials Q^* and Q if there exist numbers $0 \leq \alpha_n < \pi$ with

$$\lambda_n^2 \in \sigma(Q^*,\alpha_n) \cap \sigma(Q,\alpha_n) \quad \forall n. \tag{1.11}$$

In other words, the boundary conditions can be different for every eigenvalue λ_n^2.

Remark. By the above setting every negative value $\lambda_n^2 < 0$ can be considered as an "eigenvalue" of Q if we define α_n correspondingly. However Q and λ_n^2 define α_n, and hence (1.11) contains real information, namely that the parameter α_n is the same for Q^* and Q. This idea is useful since it is intimately connected with the problem of recovering the potential from phase shifts; see the end of Section 3 below.

Theorem 1.4. *Let $Q \in L_1(0,\infty)$ and consider the different numbers $\lambda_n = ik_n$,* $\inf k_n > 0$*. If*

$$\sum \frac{1}{k_n} = \infty, \tag{1.12}$$

then the eigenvalues λ_n^2 determine Q, i.e. there are no other potentials $Q^* \in L_1(0,\infty)$ such that the λ_n^2 are common eigenvalues of Q^* and Q in the above-defined sense.

Theorem 1.5. *For $0 < \delta$ define*

$$C_\delta = \{Q : \|Q\|_\delta = \int_0^\infty |Q(x)| e^{\delta x} dx < \infty\}.$$

Consider the numbers $\lambda_n = ik_n$, $\inf k_n > 0$*. If*

$$\sum \frac{1}{k_n} < \infty, \tag{1.13}$$

then the eigenvalues $\lambda_n^2 = -k_n^2$ do not determine the potential in C_δ, i.e. for every $Q \in C_\delta$ there exists $Q^* \in C_\delta$, $Q^* \neq Q$ such that the λ_n^2 are common eigenvalues of Q^* and Q.

Remark. As we shall check by a Liouville transformation, the statements in Theorems 1.4 and 1.5 are special cases of Theorems 1.2 and 1.3, respectively.

Remark. Condition (1.6) is necessary and sufficient for the system $\{r^{\lambda_n}\}$ to be closed in $L_1(0,a)$. Analogously (1.12) holds if and only if $\{e^{-k_n x}\}$ is closed in $L_1(0,\infty)$; see Lemma 3.2 below.

The statements corresponding to Theorems 1.4 and 1.5 for Schrödinger operators over a finite interval have been obtained earlier by the author. Namely, define $\sigma(Q,\alpha)$ as the spectrum of the operator

$$-y'' + Q(x)y = \lambda^2 y, \quad x \in [0,\pi], \tag{1.14}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \ y(\pi) = 0. \tag{1.15}$$

We say that the values λ_n^2 are common eigenvalues of the potentials Q^* and Q if there are numbers $0 \le \alpha_n < \pi$ satisfying

$$\lambda_n^2 \in \sigma(Q^*, \alpha_n) \cap \sigma(Q, \alpha_n). \tag{1.16}$$

The potential $Q \in L_1(0,\pi)$ is said to be determined by the eigenvalues $\lambda_n^2 \in \mathbf{R}$ if there is no other potential $Q^* \in L_1(0,\pi)$ satisfying (1.16) for all n.

Theorem 1.6 ([5])**.** *Let λ_n be arbitrary real numbers satisfying $\lambda_n \not\to -\infty$. The potential $Q \in L_1(0,\pi)$ is defined by the eigenvalues λ_n^2 if and only if the system $e(\Lambda) = \{e^{\pm 2i\mu x}, e^{\pm 2i\lambda_n x} : n \ge 1\}$ is closed in $L_1(0,2\pi)$; here $\mu \neq \pm\lambda_n$ is an arbitrary number.*

Let $Q \in L_1^{\mathrm{loc}}(0,\infty)$ be a potential which is a limit point at infinity, and for $\Im\lambda > 0$ let $y_1(x,\lambda)$ be the (essentially unique) solution of $-y'' + Qy = \lambda^2 y$, $0 \neq y \in L_2(0,\infty)$. Define the *m-function* of Q by

$$m(\lambda) = \frac{y_1'(0,\lambda)}{y_1(0,\lambda)}, \quad \Im\lambda > 0.$$

As is well known, the m-function defines the potential.

Theorem 1.7 (Borg [1], Marchenko [8])**.** *If the m-functions of Q and Q^* are the same, $m(Q^*, \lambda) \equiv m(Q, \lambda)$, then $Q^* = Q$ a.e.*

Using the notion of m-functions we can generalize Theorems 1.4 and 1.5 a bit further in the following form.

Theorem 1.8. *Let λ_n be arbitrary different complex numbers with $\Im\lambda_n \geq \gamma > 0$ and suppose that the m-function $m(\lambda)$ is generated by a potential $Q \in L_1(0, \infty)$. If*

$$\sum \frac{\Im\lambda_n}{|\lambda_n|^2} = \infty, \tag{1.17}$$

then the values $m(\lambda_n)$ define the m-function (and the potential). In other words, if $m(Q^*, \lambda_n) = m(Q, \lambda_n)\, \forall n$ (allowing that both sides be infinite), then $m(Q^*, \lambda) \equiv m(Q, \lambda)$ and $Q^* = Q$ a.e.

Theorem 1.9. *Let $\delta > 0$ and consider different numbers λ_n, $\Im\lambda_n \geq \gamma > 0$. The values $m(\lambda_n)$ of the m-function generated by $Q \in C_\delta$ defines $m(\lambda)$ if and only if (1.17) holds. In other words, if (1.17) is not valid, then for every $Q \in C_\delta$ there exists a different $Q^* \in C_\delta$ satisfying $m(Q^*, \lambda_n) = m(Q, \lambda_n)\, \forall n$.*

For related results on finite intervals see [5].

Remark. Theorems 1.8 and 1.2, respectively 1.9 and 1.3 are identical; see (3.23) below.

In the proofs below we borrow some ideas and methods from [5]. Section 2 is devoted to collect the preliminary material needed, while Section 3 contains the proofs of Theorems 1.2 to 1.9.

2. Preliminaries

We first recall the following known result.

Theorem 2.1 (Berezin, Shubin [3])**.** *Let $Q \in L_1(0, \infty)$ and $\lambda \in \mathbf{C} \setminus \{0\}$. Then the equation $-y'' + Qy = \lambda^2 y$ has two solutions,*

$$y_1(x) = e^{i\lambda x}(1 + \mathbf{o}(1)), \quad y_2(x) = e^{-i\lambda x}(1 + \mathbf{o}(1)), \quad x \to \infty. \tag{2.1}$$

If $\Im\lambda \geq 0$, then the solution y_1 "regular at infinity" satisfies the integral equation

$$y_1(x) = e^{i\lambda x} + \int_x^\infty \frac{\sin\lambda(t - x)}{\lambda} Q(t) y_1(t)\, dt. \tag{2.2}$$

The function $y_1(x) = y_1(x, \lambda)$ is holomorphic in $\lambda \in \mathbf{C}^+$.

Consider two potentials Q^*, $Q \in L_1(0, \infty)$, and denote by $y_1^*(x, \lambda)$, $y_1(x, \lambda)$ the corresponding y_1-solutions. Define the functions

$$F(x, \lambda) = y_1'(x, \lambda) y_1^*(x, \lambda) - {y_1^*}'(x, \lambda) y_1(x, \lambda), \tag{2.3}$$

$$F(\lambda) = F(0, \lambda). \tag{2.4}$$

By (1.11) the common eigenvalues of Q^* and Q are precisely the values $-k^2$, $k > 0$, where $F(ik) = 0$. The analogous functions in the case of finite intervals are used e.g. in Gesztesy and Simon [4].

Lemma 2.2.

$$F(\lambda) = \int_0^\infty (Q^*(x) - Q(x)) y_1^*(x,\lambda) y_1(x,\lambda)\, dx \text{ for } \Im\lambda \geq 0,\ \lambda \neq 0. \tag{2.5}$$

Proof. (2.2) implies

$$y_1'(x,\lambda) = i\lambda e^{i\lambda x} - \int_x^\infty \cos\lambda(t-x) Q(t) y_1(t,\lambda)\, dt, \tag{2.6}$$

and here the integral can be estimated by

$$\int_x^\infty \mathbf{O}\left(e^{\Im\lambda(t-x)}|Q(t)|e^{-\Im\lambda t}\right) dt = e^{-\Im\lambda x}\mathbf{O}\left(\int_x^\infty |Q|\right).$$

Consequently (for fixed λ)

$$y_1'(x,\lambda) = i\lambda e^{i\lambda x}(1 + \mathbf{o}(1)), \quad x \to \infty. \tag{2.7}$$

Comparing this with (2.1) and (2.3) gives

$$F(x,\lambda) \to 0 \quad x \to \infty,\ \lambda \text{ fixed},\ \Im\lambda \geq 0,\ \lambda \neq 0. \tag{2.8}$$

Now we have

$$\begin{aligned} F'(x,\lambda) &= y_1''(x,\lambda) y_1^*(x,\lambda) - y_1(x,\lambda) y_1^{*\prime\prime}(x,\lambda) \\ &= (Q(x) - Q^*(x)) y_1^*(x,\lambda) y_1(x,\lambda), \end{aligned}$$

whence

$$F(\lambda) - F(N,\lambda) = \int_0^N (Q^*(x) - Q(x)) y_1^*(x,\lambda) y_1(x,\lambda)\, dx.$$

Taking the limit $N \to \infty$ gives (2.5). □

Lemma 2.3. *$F(\lambda)$ is bounded in every half-plane $\{\lambda : \Im\lambda \geq \gamma > 0\}$.*

Proof. The function

$$z(x,\lambda) = y_1(x,\lambda) e^{-i\lambda x} = 1 + \mathbf{o}(1)$$

is bounded for $x \geq 0$ by a bound depending on λ. From

$$z(x,\lambda) = 1 + \int_x^\infty \frac{e^{2i\lambda(t-x)} - 1}{2i\lambda} Q(t) z(t,\lambda)\, dt$$

we obtain

$$|z(x,\lambda)| \leq 1 + \frac{\|z\|_\infty}{|\lambda|}\int_x^\infty |Q|,$$

i.e.

$$\|z\|_\infty \leq 1 + \frac{\int_0^\infty |Q|}{|\lambda|}\|z\|_\infty.$$

This means that $\|z\|_\infty \leq 2$ if $|\lambda| \geq 2\int_0^\infty |Q|$, or

$$|y_1(x,\lambda)| \leq 2e^{-\Im\lambda x} \text{ if } |\lambda| \geq 2\int_0^\infty |Q|,\ \Im\lambda \geq 0. \tag{2.9}$$

Consequently from (2.5)

$$|F(\lambda)| \leq 4\int_0^\infty |Q^* - Q| \text{ if } |\lambda| \geq 2\max\left(\int_0^\infty |Q|, \int_0^\infty |Q|\right),\ \Im\lambda \geq 0.$$

This estimate with the continuity of F shows its boundedness on $\Im\lambda \geq \gamma$. □

From now on we consider more rapidly decaying potentials satisfying

$$\int_0^\infty x|Q(x)|\,dx < \infty. \tag{2.10}$$

This means that the function

$$\sigma(x) = \int_x^\infty |Q|$$

belongs to $L_1(0,\infty)$; indeed,

$$\int_0^\infty \sigma(x)\,dx = \int_0^\infty \int_x^\infty |Q(t)|\,dt\,dx = \int_0^\infty t|Q(t)|\,dt.$$

Define

$$\sigma_1(x) = \int_x^\infty \sigma.$$

We need the following well-known facts.

Lemma 2.4 (Marchenko [7], Chapter III)**.** *Suppose* (2.10)*. Then there exists a function* $K(x,t)$*, continuous for* $0 \le x \le t < \infty$*, satisfying the properties*

$$y_1(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt \quad \text{for} \quad x \ge 0,\, \Im\lambda \ge 0, \tag{2.11}$$

$$K(x,x) = 1/2\int_x^\infty Q, \tag{2.12}$$

$$|K(x,t)| \le 1/2\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)-\sigma_1(\frac{x+t}{2})}. \tag{2.13}$$

Define further

$$H(u,v) = K(u-v,u+v), \quad 0 \le v \le u < \infty. \tag{2.14}$$

Then

$$H(u,v) = \sum_{n=0}^\infty H_n(u,v), \tag{2.15}$$

where

$$H_0(u,v) = 1/2\int_u^\infty Q, \tag{2.16}$$

$$H_n(u,v) = \int_u^\infty \int_0^v Q(\alpha-\beta)H_{n-1}(\alpha,\beta)\,d\beta\,d\alpha, \;\; n \ge 1. \tag{2.17}$$

Finally we have

$$|H_n(u,v)| \le 1/2\sigma(u)\frac{[\sigma_1(u-v)-\sigma_1(v)]^n}{n!}, \quad n \ge 0. \tag{2.18}$$

Let Q, $Q^* \in C_\delta$ (as in Theorem 1.5) and define the kernels $K(x,t)$, $K^*(x,t)$ corresponding to Lemma 2.4. Introduce the third kernel

$$\begin{aligned} K_1(x,t,Q,Q^*) &= 2K(t,2x-t) + 2K^*(t,2x-t) \\ &\quad + 2\int_t^{2x-t} K(t,u)K^*(t,2x-u)\,du, \quad 0 \le t \le x < \infty, \end{aligned} \tag{2.19}$$

and the corresponding Volterra-type integral operator

$$A_{Q^*} : C_\delta \to C_\delta,$$
$$(A_{Q^*}h)(x) = h(x) + \int_0^x K_1(x,t,Q,Q^*)h(t)\,dt.$$

Its relevance is justified by the following formula.

Lemma 2.5.

$$F(\lambda) = \int_0^\infty e^{2i\lambda x}\left[A_{Q^*}(Q^*-Q)\right](x)\,dx. \tag{2.20}$$

Proof. It is a simple calculation by substituting (2.11) into (2.5):

$$F(\lambda) = \int_0^\infty (Q^*(x)-Q(x))\left[e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt\right]$$
$$\cdot \left[e^{i\lambda x} + \int_x^\infty K^*(x,t)e^{i\lambda t}dt\right]\,dx.$$

By interchanging the order of integrations we get

$$\begin{aligned}
&\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty K(x,t)e^{i\lambda(x+t)}dt\,dx\\
&= 2\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty K(x,2\tau-x)e^{2i\lambda\tau}d\tau\,dx\\
&= 2\int_0^\infty e^{2i\lambda\tau}\int_0^\tau (Q^*(x)-Q(x))K(x,2\tau-x)\,dx\,d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty K(x,t)\int_x^\infty K^*(x,\tau)e^{i\lambda(t+\tau)}d\tau\,dt\,dx\\
&= 2\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty K(x,t)\int_{\frac{x+t}{2}}^\infty K^*(x,2u-t)e^{2i\lambda u}du\,dt\,dx\\
&= 2\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty e^{2i\lambda u}\int_x^{2u-x} K(x,t)K^*(x,2u-t)\,dt\,du\,dx\\
&= 2\int_0^\infty e^{2i\lambda u}\int_0^u (Q^*(x)-Q(x))\int_x^{2u-x} K(x,t)K^*(x,2u-t)\,dt\,dx\,du.
\end{aligned}$$

We used the fact that $\int_x^\infty |K(x,t)|\,dt$ is bounded in x; see (2.13). Finally we get

$$\begin{aligned}
F(\lambda) = \int_0^\infty e^{2i\lambda x}\Big[&Q^*(x)-Q(x) + 2\int_0^x (Q^*(t)-Q(t))K(t,2x-t)\,dt\\
&+ 2\int_0^x (Q^*(t)-Q(t))K^*(t,2x-t)\,dt\\
&+ 2\int_0^x (Q^*(t)-Q(t))\int_t^{2x-t} K(t,u)K^*(t,2x-u)\,du\,dt\Big]\,dx,
\end{aligned}$$

which is (2.20). □

Define the kernel

$$\tilde{K}(x,t,Q,Q^*) = e^{\delta(x-t)}K_1(x,t,Q,Q^*)$$

and the corresponding Volterra operator

$$\tilde{K} : L_1(0,\infty) \to L_1(0,\infty),$$
$$(\tilde{K}h)(x) = \int_0^x \tilde{K}(x,t,Q,Q^*)h(t)\,dt.$$

Lemma 2.6. *Let $Q, Q^* \in C_\delta$ for some $\delta > 0$. Then*
a) *$\tilde{K} : L_1(0,\infty) \to L_1(0,\infty)$ is compact, and*
b) *$A_{Q^*} : C_\delta \to C_\delta$ is an (onto) isomorphism.*

Proof. From

$$\sigma_1(x) = \int_x^\infty \int_t^\infty |Q(\tau)|\,d\tau\,dt \le \int_x^\infty e^{-\delta t} \int_t^\infty e^{\delta\tau}|Q(\tau)|\,d\tau\,dt$$
$$\le \|Q\|_\delta \int_x^\infty e^{-\delta t}dt = \frac{1}{\delta}e^{-\delta x}\|Q\|_\delta$$

and from (2.13) we get

$$|K(x,t)| \le c(D,\delta)\int_{\frac{x+t}{2}}^\infty |Q| \quad \text{if} \quad \|Q\|_\delta \le D. \tag{2.21}$$

Consequently

$$|K_1(x,t,Q,Q^*)| \le c(D,\delta)\Bigg[\int_x^\infty (|Q^*|+|Q|) + \int_t^{2x-t}\left(\int_{\frac{t+u}{2}}^\infty |Q|\right)\left(\int_{x+\frac{t-u}{2}}^\infty |Q^*|\right)du\Bigg] \quad \text{if} \quad \|Q^*\|_\delta, \|Q\|_\delta \le D, \tag{2.22}$$

and a corresponding bound can be given for $\tilde{K}$ after the multiplication by $e^{\delta(x-t)}$.

The operator norm of $\tilde{K}$ has the bound

$$\|\tilde{K}\|_{1,1} \le \sup_t \int_t^\infty |\tilde{K}(x,t,Q,Q^*)|\,dx \tag{2.23}$$

as it is seen from

$$\int_0^\infty \left|\int_0^x \tilde{K}(x,t)h(t)\,dt\right|dx \le \int_0^\infty |h(t)| \int_t^\infty |\tilde{K}(x,t)|\,dx\,dt$$

(the dependence of $\tilde{K}$ on Q and Q^* is not indicated).

In proving a) we will approximate $\tilde{K}$ in operator norm by operators of finite-dimensional range. First let

$$\tilde{K}_N(x,t) = \begin{cases} \tilde{K}(x,t) & \text{for} \quad x \le N, \\ 0 & \text{for} \quad x > N. \end{cases}$$

We will check that

$$\|\tilde{K} - \tilde{K}_N\|_{1,1} \to 0 \quad \text{if} \quad N \to \infty. \tag{2.24}$$

Indeed,

$$\sup_t \int_t^\infty \left|\tilde{K}(x,t) - \tilde{K}_N(x,t)\right| dx = \max(I_1, I_2),$$

where

$$I_1 = \sup_{0\le t\le N} \int_N^\infty |\tilde{K}(x,t)|\,dx,$$
$$I_2 = \sup_{t>N} \int_t^\infty |\tilde{K}(x,t)|\,dx.$$

In I_1 we have by (2.22)

$$\int_N^\infty e^{\delta(x-t)} \int_x^\infty |Q(\tau)|\,d\tau\,dx = \int_N^\infty |Q(\tau)| \int_N^\tau e^{\delta(x-t)} dx\,d\tau \tag{2.25}$$
$$\le 1/\delta e^{-\delta t} \int_N^\infty |Q(\tau)| e^{\delta\tau} d\tau \to 0 \quad \text{if} \quad N\to\infty$$

and

$$\int_N^\infty e^{\delta(x-t)} \int_t^{2x-t} \left(\int_{\frac{t+u}{2}}^\infty |Q|\right)\left(\int_{x+\frac{t-u}{2}}^\infty |Q^*|\right) du\,dx \tag{2.26}$$
$$= \int_t^\infty \left(\int_{\frac{t+u}{2}}^\infty |Q|\right) \int_{\max(N,\frac{u+t}{2})}^\infty e^{\delta(x-t)} \int_{x+\frac{t-u}{2}}^\infty |Q^*(\tau)|\,d\tau\,dx\,du$$
$$= \int_t^\infty \left(\int_{\frac{t+u}{2}}^\infty |Q|\right) \int_{\max(N+\frac{t-u}{2},t)}^\infty |Q^*(\tau)| \int_{\max(N,\frac{t+u}{2})}^{\tau+\frac{u-t}{2}} e^{\delta(x-t)} dx\,d\tau\,du$$
$$\le 1/\delta \int_t^\infty \left(\int_{\frac{t+u}{2}}^\infty |Q|\right) \int_{\max(N+\frac{t-u}{2},t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau \cdot e^{-3/2\delta t} e^{\delta u/2} du$$
$$= 1/\delta \int_t^\infty |Q(\nu)| \int_t^{2\nu-t} e^{\delta u/2} \int_{\max(N+\frac{t-u}{2},t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau\,du\,d\nu \cdot e^{-3/2\delta t}$$
$$\le 1/\delta e^{-3/2\delta t} \int_t^\infty |Q(\nu)| \int_{\max(N+t-\nu,t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau \cdot \int_t^{2\nu-t} e^{\delta u/2} du\,d\nu$$
$$\le 2/\delta^2 e^{-2\delta t} \int_t^\infty |Q(\nu)| e^{\delta\nu} \int_{\max(N+t-\nu,t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau\,d\nu.$$

This expression is small (uniformly in t) if N is large. Indeed, if $\max(N+t-\nu,t)\ge N/2$, then the inner integral is small; if $\max(N+t-\nu,t) < N/2$, then $\nu > N/2$ and on this domain the outer integral is small. This verifies that $I_1 \to 0$ if $N\to\infty$. In I_2 we can apply similar manipulations (with t instead of N), namely for $N\to\infty$

$$\int_t^\infty e^{\delta(x-t)} \left(\int_x^\infty |Q|\right) dx \le 1/\delta e^{-\delta t} \int_t^\infty |Q(\tau)| e^{\delta\tau} d\tau$$
$$\le 1/\delta e^{-\delta N} \|Q\|_\delta \to 0,$$

$$\int_t^\infty e^{\delta(x-t)} \int_t^{2x-t} \left(\int_{\frac{t+u}{2}}^\infty |Q|\right)\left(\int_{x+\frac{t-u}{2}}^\infty |Q^*|\right) du\,dx$$
$$\le 2/\delta^2 e^{-2\delta t} \int_t^\infty |Q(\nu)| e^{\delta\nu} \int_t^\infty |Q^*(\tau)| e^{\delta\tau} d\tau\,d\nu$$
$$\le 2/\delta^2 e^{-2\delta N} \|Q\|_\delta \|Q^*\|_\delta \to \infty.$$

Thus (2.24) is verified.

We extend the kernel $\tilde{K}_N$ by zero to $[0,N]^2$ and (for fixed N and large M) divide $[0,N]^2$ into M^2 small squares of edge length N/M. In every small square define $\tilde{K}_{N,M}$ to be constant, namely the value of $\tilde{K}_N$ at the left upper vertex. Since $\tilde{K}_N$ is continuous on the compact set $0 \le t \le x \le N$, $\tilde{K}_N - \tilde{K}_{N,M}$ is uniformly small for large M, except for the small squares along the diagonal $x = t$. In these exceptional squares $\tilde{K}_N - \tilde{K}_{N,M} = \tilde{K}_N$ is bounded (by a bound depending on N). Consequently

$$\sup_{t \le N} \int_t^\infty |\tilde{K}_N(x,t) - \tilde{K}_{N,M}(x,t)|\, dx \to 0, \quad M \to \infty.$$

With (2.24) this means that $\tilde{K}$ can be approximated by the $\tilde{K}_{N,M}$ in operator norm. Since the $\tilde{K}_{N,M}$ have finite-dimensional range, $\tilde{K}$ is compact, so statement a) is proved. Point b) is a corollary of a). Indeed, a) implies the compactness of the integral operator with kernel K_1. We know $A_{Q^*} = I + K_1$. If A_{Q^*} is not an isomorphism, then -1 must be an eigenvalue of K_1, i.e. there exists $0 \ne h \in C_\delta$ such that $-h = K_1 h$. But this is impossible for a Volterra operator with a continuous kernel. So A_{Q^*} is an isomorphism as asserted. □

Lemma 2.7. *Let $Q, Q^*, Q^{**} \in C_\delta$ for some $\delta > 0$ and suppose that $\|Q\|_\delta, \|Q^*\|_\delta$, $\|Q^{**}\|_\delta \le D$. Then*

$$\|(A_{Q^{**}} - A_{Q^*})h\|_\delta \le c(D,\delta)\|Q^{**} - Q^*\|_\delta \|h\|_\delta, \quad \forall\, h \in C_\delta. \tag{2.27}$$

The constant $c(D,\delta)$ depends only on its arguments.

Proof. Consider the functions $H_n(u,v)$ defined in Lemma 2.4. Denote

$$\varrho(u,v) = \sigma_1(u-v) - \sigma_1(v), \quad 0 \le v \le u;$$

it is increasing in v for fixed u and decreasing in u for fixed v since

$$\frac{\partial \varrho}{\partial u} = -\int_{u-v}^{u} |Q| < 0.$$

Analogously define ϱ^* with Q^* instead of Q. Finally let

$$\tilde{\sigma}(u) = \int_u^\infty |Q^* - Q|, \quad \tilde{\sigma}_1(u) = \int_u^\infty \tilde{\sigma}, \quad \tilde{\varrho}(u,v) = \tilde{\sigma}_1(u-v) - \tilde{\sigma}_1(u).$$

The proof of (2.27) is based on the estimate

$$|H_n^*(u,v) - H_n(u,v)| \le 1/2\sigma^*(u)\tilde{\varrho}(u,v)\frac{[\varrho(u,v) + \varrho^*(u,v)]^{n-1}}{(n-1)!} + 1/2\tilde{\sigma}(u)\frac{\varrho(u,v)^n}{n!}; \tag{2.28}$$

for $n = 0$ only the second summand is considered on the right. We apply induction on n. For $n = 0$

$$|H_0^*(u,v) - H_0(u,v)| = 1/2\left|\int_u^\infty (Q^* - Q)\right| \le 1/2\tilde{\sigma}(u).$$

Suppose (2.28) for a value of n. Then

$$\begin{aligned}(2.29)\qquad & H^*_{n+1}(u,v) - H_{n+1}(u,v) \\ &= \int_u^\infty \int_0^v \left(Q^*(\alpha-\beta) - Q(\alpha-\beta)\right) H^*_n(\alpha,\beta)\, d\beta\, d\alpha \\ &+ \int_u^\infty \int_0^v Q(\alpha-\beta)\left(H^*_n(\alpha,\beta) - H_n(\alpha,\beta)\right) d\beta\, d\alpha \overset{def}{=} I_1 + I_2.\end{aligned}$$

We use the identity

$$\begin{aligned}(2.30)\qquad & \int_u^\infty \int_0^v \left|Q^*(\alpha-\beta) - Q(\alpha-\beta)\right| d\beta\, d\alpha \\ &= \int_u^\infty \left(\tilde\sigma(\alpha-v) - \tilde\sigma(\alpha)\right) d\alpha = \tilde\varrho(u,v)\end{aligned}$$

and (2.18) to obtain

$$\begin{aligned}(2.31)\qquad |I_1| &\le 1/2 \int_u^\infty \int_0^v \left|Q^*(\alpha-\beta) - Q(\alpha-\beta)\right| \sigma^*(\alpha) \frac{\varrho^*(\alpha,\beta)^n}{n!}\, d\beta\, d\alpha \\ &\le 1/2\sigma^*(u) \frac{\varrho^*(u,v)^n}{n!} \int_u^\infty \int_0^v \left|Q^*(\alpha-\beta) - Q(\alpha-\beta)\right| d\beta\, d\alpha \\ &= 1/2\sigma^*(u)\tilde\varrho(u,v) \frac{\varrho^*(u,v)^n}{n!}.\end{aligned}$$

On the other hand

$$\begin{aligned}(2.32)\qquad & |I_2| \le 1/2 \int_u^\infty \int_0^v |Q(\alpha-\beta)| \cdot \tilde\sigma(\alpha) \frac{\varrho(\alpha,\beta)^n}{n!} d\beta\, d\alpha \\ &+1/2 \int_u^\infty \int_0^v |Q(\alpha-\beta)| \cdot \sigma^*(\alpha)\tilde\varrho(\alpha,\beta) \frac{[\varrho(\alpha,\beta) + \varrho^*(\alpha,\beta)]^{n-1}}{(n-1)!} d\beta\, d\alpha \\ &\overset{def}{=} I_{21} + I_{22}.\end{aligned}$$

Apply the identity

$$\int_0^v |Q(\alpha-\beta)|\, d\beta = -\frac{\partial\varrho}{\partial\alpha}(\alpha,v)$$

to get

$$\begin{aligned}(2.33)\qquad I_{21} &\le 1/2\tilde\sigma(u) \int_u^\infty \frac{\varrho(\alpha,v)^n}{n!} \int_0^v |Q(\alpha-\beta)|\, d\beta\, d\alpha \\ &= 1/2\tilde\sigma(u) \frac{\varrho(u,v)^{n+1}}{(n+1)!},\end{aligned}$$

$$\begin{aligned}(2.34)\qquad I_{22} &\le 1/2\sigma^*(u)\tilde\varrho(u,v) \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(n-1)!} \varrho^*(u,v)^{n-1-k} \\ &\cdot \int_u^\infty \varrho(\alpha,v)^k \int_0^v |Q(\alpha-\beta)|\, d\beta\, d\alpha \\ &= 1/2\sigma^*(u)\tilde\varrho(u,v) \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(n-1)!} \frac{1}{k+1} \varrho^*(u,v)^{n-1-k} \varrho(u,v)^{k+1} \\ &= 1/2\sigma^*(u)\tilde\varrho(u,v) \sum_{k=1}^{n} \frac{\binom{n}{k}}{n!} \varrho^*(u,v)^{n-k} \varrho(u,v)^k.\end{aligned}$$

Summing up (2.31)-(2.34) gives

$$|H^*_{n+1}(u,v) - H_{n+1}(u,v)| \le 1/2\sigma^*(u)\tilde{\varrho}(u,v)\frac{[\varrho(u,v)+\varrho^*(u,v)]^n}{n!} + 1/2\tilde{\sigma}(u)\frac{\varrho(u,v)^{n+1}}{(n+1)!}$$

which verifies (2.28) for $n+1$. Consequently

$$|H^*(u,v) - H(u,v)| \le 1/2\sigma^*(u)\tilde{\varrho}(u,v)e^{\varrho(u,v)+\varrho^*(u,v)} + 1/2\tilde{\sigma}(u)e^{\varrho(u,v)}. \tag{2.35}$$

Since we have

$$\tilde{\varrho}(u,v) = \tilde{\sigma}_1(u-v) - \tilde{\sigma}_1(u) \le \tilde{\sigma}_1(0) = \int_0^\infty t|Q^*(t) - Q(t)|\,dt \le c(\delta)\|Q^* - Q\|_\delta,$$

the estimate (2.35) can be continued as follows:

$$|H^*(u,v) - H(u,v)| \le c(D,\delta)\left[\int_u^\infty |Q^* - Q| + \|Q^* - Q\|_\delta \cdot \int_u^\infty |Q^*|\right] \tag{2.36}$$

or, from (2.14)

$$|K^*(x,t) - K(x,t)| \le c(D,\delta)\left[\int_{\frac{x+t}{2}}^\infty |Q^* - Q| + \|Q^* - Q\|_\delta \cdot \int_{\frac{x+t}{2}}^\infty |Q^*|\right]. \tag{2.37}$$

Now we are able to prove the estimate

$$\int_t^\infty e^{\delta x}\big|K_1(x,t,Q,Q^{**}) - K_1(x,t,Q,Q^*)\big|\,dx \le C(D,\delta)\|Q^{**} - Q^*\|_\delta \quad \text{for all} \quad t \ge 0. \tag{2.38}$$

This implies the Lipschitz property (2.27) because

$$\|A_{Q^{**}} - A_{Q^*}\| \le \sup_{t\ge 0}\int_t^\infty e^{\delta(x-t)}|K_1(x,t,Q,Q^{**}) - K_1(x,t,Q,Q^*)|\,dx.$$

Now use the decomposition (2.19) in

$$\begin{aligned} &\int_t^\infty e^{\delta x}|K_1(x,t,Q,Q^{**}) - K_1(x,t,Q,Q^*)|\,dx \\ &\le 2\int_t^\infty e^{\delta x}|K^{**}(t,2x-t) - K^*(t,2x-t)|\,dx \\ &+2\int_t^\infty e^{\delta x}\int_t^{2x-t}|K(t,u)||K^{**}(t,2x-u) - K^*(t,2x-u)|\,du\,dx \\ &\overset{def}{=} 2I_1 + 2I_2. \end{aligned} \tag{2.39}$$

We estimate I_1 using (2.37) as follows:

$$\begin{aligned}&\int_t^\infty e^{\delta x}\int_x^\infty |Q^{**}(\tau)-Q^*(\tau)|\,d\tau\,dx\\ =&\int_t^\infty |Q^{**}(\tau)-Q^*(\tau)|\int_t^\tau e^{\delta x}dx\,d\tau\\ \le& 1/\delta\int_t^\infty |Q^{**}(\tau)-Q^*(\tau)|e^{\delta\tau}d\tau = 1/\delta\|Q^{**}-Q^*\|_\delta\end{aligned}$$

and analogously

$$\|Q^{**}-Q^*\|_\delta\int_t^\infty e^{\delta x}\left(\int_x^\infty |Q^*|\right)dx \le D/\delta\|Q^{**}-Q^*\|_\delta,$$

hence

$$|I_1|\le c(D,\delta)\|Q^{**}-Q^*\|_\delta.$$

In I_2 we have, as in verifying (2.24) above,

$$\begin{aligned}&\int_t^\infty e^{\delta x}\int_t^{2x-t}|K(t,u)|\left(\int_{x+\frac{t-u}{2}}^\infty |Q^{**}-Q^*|\right)du\,dx\\ \le& c(D,\delta)\int_t^\infty e^{\delta x}\int_t^{2x-t}\left(\int_{\frac{t+u}{2}}^\infty |Q|\right)\left(\int_{x+\frac{t-u}{2}}^\infty |Q^{**}-Q^*|\right)du\,dx\\ \le& c(D,\delta)\int_t^\infty |Q(\nu)|e^{\delta\nu}\int_t^\infty |Q^*(\xi)-Q(\xi)|e^{\delta\xi}d\xi\,d\nu\\ \le& c(D,\delta)\|Q^*-Q\|_\delta,\end{aligned}$$

and by the same way we obtain

$$\begin{aligned}&\|Q^{**}-Q^*\|_\delta\cdot\int_t^\infty e^{\delta x}\int_t^{2x-t}|K(t,u)|\left(\int_{x+\frac{t-u}{2}}^\infty |Q^*|\right)du\,dx\\ &\le c(D,\delta)\|Q^{**}-Q^*\|_\delta.\end{aligned}$$

Thus we have checked the estimate (2.38). The proof of Lemma 2.7 is complete. □

3. Proof of the main results

Proof of Theorem 1.8. Suppose indirectly that (1.17) holds, however there are two potentials $Q^*, Q\in L_1(0,\infty)$ with $m(Q^*,\lambda_n)=m(Q,\lambda_n)$, $\forall n$. Introduce the function $F(\lambda)$ as in (2.4); then $F(\lambda_n)=0$, $\forall n$. Let $\mu>0$ be arbitrary; by Lemma 2.3 the function $F(\lambda+i\mu)$ is bounded analytic in the upper half-plane $\Im\lambda>0$. So if F is not identically vanishing, its zeros must satisfy the Blaschke condition

$$\sum_{F(\lambda+i\mu)=0,\Im\lambda>0}\frac{\Im\lambda}{1+|\lambda|^2}<\infty;$$

see e.g. Duren [2]. In particular, for the zeros $\lambda_n-i\mu$

$$\sum_{\Im\lambda_n>\mu}\frac{\Im\lambda_n-\mu}{1+|\lambda_n-i\mu|^2}<\infty\quad\text{ for all }\quad\mu>0.$$

We can suppose that $\gamma > 2\mu$ and then

$$\sum \frac{\Im\lambda_n}{|\lambda_n|^2} \le c(\mu) \sum \frac{\Im\lambda_n - \mu}{1 + |\lambda_n - i\mu|^2} < \infty,$$

a contradiction. Consequently $F(\lambda) \equiv 0$, i.e. $m^* \equiv m$, and by Theorem 1.7, $Q^* = Q$ a.e. □

In order to prove Theorem 1.9 we need

Lemma 3.1 ([5])**.** *Let B_1 and B_2 be Banach spaces. For every $q \in B_1$ let a continuous linear operator*

$$A_q : B_1 \to B_2$$

be defined so that for some $q_0 \in B_1$

$$A_{q_0} : B_1 \to B_2 \quad \text{is an (onto) isomorphism} \tag{3.1}$$

and the mapping $q \mapsto A_q$ is Lipschitzian in the sense that

$$\|(A_{q^*} - A_q)h\|_2 \le c(q_0)\|q^* - q\|_1 \|h\|_1 \tag{3.2}$$
$$\text{for all } h, q, q^* \in B_1 \text{ with } \|q\|_1, \|q^*\|_1 \le 2\|q_0\|_1,$$

the constant $c(q_0)$ being independent of q, q^* and h. Then the set $\{A_q(q - q_0) : q \in B_1\}$ contains a ball in B_2 with center at the origin.

The Müntz theorem about the closedness of exponential systems is known in several versions. The author did not find a proper reference containing the version formulated below, so a proof is provided.

Lemma 3.2. *Let λ_n be different complex numbers with $\Im\lambda_n \ge 0$, $\liminf \Im\lambda_n > 0$. Then*

$$\{e^{i\lambda_n x}\} \text{ is closed in } L_1(0,\infty) \Longleftrightarrow \sum \frac{\Im\lambda_n}{1 + |\lambda_n|^2} = \infty. \tag{3.3}$$

Proof. The if part: Let $h \in L_1(0,\infty)$, $\int_0^\infty h(x)e^{i\lambda_n x}dx = 0$ for all n. Define $H(z) = \int_0^\infty h(x)e^{izx}dx$, $\Im z \ge 0$. It is bounded analytic in the upper half-plane, so if $H \ne 0$, its zeros satisfy the Blaschke condition and then

$$\sum_{\Im\lambda_n > 0} \frac{\Im\lambda_n}{1 + |\lambda_n|^2} < \infty,$$

a contradiction. This implies $H \equiv 0$ and then $h = 0$, i.e. the system $\{e^{i\lambda_n x}\}$ is closed in L_1.

The only if part: Let

$$\sum_{\Im\lambda_n > 0} \frac{\Im\lambda_n}{1 + |\lambda_n|^2} < \infty.$$

Since $\liminf \lambda_n > 0$, there exists $\delta > 0$ such that $\Im\lambda_n \ge 2\delta$ with finitely many exceptions. Thus

$$\sum \frac{\Im(\lambda_n + i\delta)}{1 + |\lambda_n + i\delta|^2} < \infty. \tag{3.4}$$

This means that there exists a Blaschke product $B(z)$ with $B(\lambda_n + i\delta) = 0\ \forall n$; see [2]. The function $\frac{B(z)}{z+i}$ belongs to the Hardy space $H^2(\mathbf{C}^+)$. By the Paley-Wiener theorem there exists $0 \neq g \in L_2(0,\infty)$ with

$$\frac{B(z)}{z+i} = \int_0^\infty g(x)e^{izx}dx.$$

Consequently

$$0 = \int_0^\infty g(x)e^{i(\lambda_n+i\delta)x}dx = \int_0^\infty g(x)e^{-\delta x}e^{i\lambda_n x}dx = \int_0^\infty h(x)e^{i\lambda_n x}dx$$

with

$$0 \neq h(x) = g(x)e^{-\delta x} \in L_1(0,\infty).$$

Thus $\{e^{i\lambda_n x}\}$ is not closed in L_1. □

Proof of Theorem 1.9. Let $Q \in C_\delta$ be arbitrary; our task is to find a different $Q^* \in C_\delta$ with $m(Q^*,\lambda_n) = m(Q,\lambda_n)$, i.e. $F(\lambda_n) = 0$. Suppose that (1.17) is not true. Then

$$\sum \frac{\Im(2\lambda_n + i\delta)}{|2\lambda_n + i\delta|^2} < \infty \tag{3.5}$$

(because $\Im\lambda_n \geq \gamma > 0$). By Lemma 3.2 there exists $0 \neq g \in L_1(0,\infty)$ with

$$0 = \int_0^\infty g(x)e^{(2i\lambda_n-\delta)x}dx = \int_0^\infty h(x)e^{2i\lambda_n x}dx, \tag{3.6}$$

$$h(x) = g(x)e^{-\delta x} \in C_\delta. \tag{3.7}$$

Recall from Lemma 2.5 the identity

$$F(\lambda) = \int_0^\infty e^{2i\lambda x}\left[A_{Q^*}(Q^* - Q)\right](x)\,dx.$$

Comparing this with (3.6) we see that $F(\lambda_n) = 0$ is guaranteed if

$$A_{Q^*}(Q^* - Q) = ch \quad \text{for some } c \neq 0. \tag{3.8}$$

To check (3.8) we apply Lemma 3.1 with $B_1 = B_2 = C_\delta$. The conditions (3.1) and (3.2) are verified in Lemmas 2.6 and 2.7, respectively. Since the set $\{A_{Q^*}(Q^* - Q) : Q^* \in C_\delta\}$ contains all elements of C_δ of sufficiently small norm, there exists a potential $Q^* \in C_\delta$, $Q^* \neq Q$ satisfying (3.8). Now from $F(\lambda_n) = 0$ it follows that $m^*(\lambda_n) = m(\lambda_n)$, though $m^* \neq m$. □

Proof of Theorems 1.4 *and* 1.5. If $\lambda_n = ik_n$, $\inf k_n = \gamma > 0$, then $m^*(\lambda_n) = m(\lambda_n)$ if and only if the values $\lambda_n^2 = -k_n^2$ are common eigenvalues of Q^* and Q. Since $\Im\lambda_n/|\lambda_n|^2 = 1/k_n$, Theorems 1.4 and 1.5 are contained in Theorems 1.8 and 1.9, respectively. □

Proof of Theorems 1.2 *and* 1.3. Let $Q \in L_1(0,\infty)$ and consider the solution $y_1(x,\lambda) = e^{i\lambda x}(1 + \mathbf{o}(1))$ of $-y'' + Qy = \lambda^2 y$ with $\Im\lambda \geq 0$. Introduce the function

$$\varphi(r,\lambda) = \sqrt{r}y_1(\ln \tfrac{a}{r}, i\lambda) \quad \text{for } 0 < r \leq a,\ \Re\lambda \geq 0. \tag{3.9}$$

A short calculation gives that

$$\varphi''(r,\lambda) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r,\lambda) - 1/r^2 Q(\ln \tfrac{a}{r})\varphi(r,\lambda) = 0,\ 0 < r \leq a. \tag{3.10}$$

This means that φ satisfies (1.1) (for $r \le a$) if

$$q(r) = 1/r^2 Q(\ln \tfrac{a}{r}) + 1. \tag{3.11}$$

We observe that

$$rq(r) \in L_1(0,a) \Longleftrightarrow Q(x) \in L_1(0,\infty). \tag{3.12}$$

Indeed, substituting $x = \ln \frac{a}{r}$ gives

$$\int_0^\infty |Q(x)|\,dx = \int_0^a \frac{1}{r}|Q(\ln \tfrac{a}{r})|\,dr = \int_0^a r|q(r)-1|\,dr,$$

and this is finite if and only if $rq(r) \in L_1(0,a)$. Analogously for $0 < \delta < 2$

$$r^{1-\delta} q(r) \in L_1(0,a) \Longleftrightarrow Q \in C_\delta \tag{3.13}$$

since

$$\int_0^\infty |Q(x)| e^{\delta x} dx = a^\delta \int_0^a r^{1-\delta} |q(r)-1|\,dr.$$

From $y_1(x,\lambda) = e^{i\lambda x}(1+\mathbf{o}(1))$ we infer

$$\varphi(r,\lambda) = a^{-\lambda} r^{\lambda+1/2}(1+\mathbf{o}(1)), \quad r \to 0+, \tag{3.14}$$

which corresponds to (1.2).

In the potential-free domain $r \ge a$ (1.1) has the form

$$\varphi'' + \varphi = \frac{\lambda^2 - 1/4}{r^2}\varphi, \quad r \ge a. \tag{3.15}$$

Its general solution can be given by Bessel functions

$$\varphi(r,\lambda) = c(\lambda)\sqrt{r}\left[\cos\alpha(\lambda)\cdot J_\lambda(r) + \sin\alpha(\lambda)\cdot Y_\lambda(r)\right], \tag{3.16}$$

with $\alpha(\lambda) \in \mathbf{C}$; see Watson [10]. Taking into account the asymptotic forms

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \nu\pi/2 - \pi/4) + \mathbf{O}(r^{-3/2}), \quad r \to \infty, \tag{3.17}$$

$$Y_\nu(r) = \sqrt{\frac{2}{\pi r}} \sin(r - \nu\pi/2 - \pi/4) + \mathbf{O}(r^{-3/2}), \quad r \to \infty, \tag{3.18}$$

we get

$$\begin{aligned} \varphi(r,\lambda) &= c(\lambda)\sqrt{\frac{2}{\pi}} \cos(r - \lambda\pi/2 - \pi/4 - \alpha(\lambda)) + \mathbf{O}(1/r) \\ &= c(\lambda)\sqrt{\frac{2}{\pi}} \sin(r - \pi/2(\lambda - 1/2) - \alpha(\lambda)) + \mathbf{O}(1/r), \end{aligned} \tag{3.19}$$

which is compatible with (1.3) if and only if

$$\delta(\lambda) \equiv -\alpha(\lambda) \pmod{\pi}. \tag{3.20}$$

On the other hand,

$$\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)} = \frac{\cos\alpha(\lambda)\frac{d(\sqrt{r}J_\lambda(r))}{dr}(a) + \sin\alpha(\lambda)\frac{d(\sqrt{r}Y_\lambda(r))}{dr}(a)}{\sqrt{a}\left[\cos\alpha(\lambda)J_\lambda(a) + \sin\alpha(\lambda)Y_\lambda(a)\right]}. \tag{3.21}$$

Finally from (3.9) it follows that

$$\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)} = \frac{1/2a^{-1/2}y_1(0,i\lambda) - a^{-1/2}y_1'(0,i\lambda)}{a^{1/2}y_1(0,i\lambda)}. \tag{3.22}$$

Now (3.20)-(3.22) show that the knowledge of the shift $\delta(\lambda)$ (for a fixed λ, $\Re\lambda \geq 0$) means the knowledge of $\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)}$ and of $\frac{y_1'(0,i\lambda)}{y_1(0,i\lambda)}$. In other words for Q, $Q^* \in L_1(0,\infty)$ and for q, q^* defined by (3.11), we have

$$\delta(q^*,\lambda) \equiv \delta(q,\lambda) (\mathrm{mod}\, \pi) \iff m(Q^*, i\lambda) = m(Q, i\lambda) \tag{3.23}$$

for the corresponding m-functions. Taking into account (3.12) and (3.13) we see that Theorem 1.2 and 1.8, respectively Theorem 1.3 and 1.9, are the same statements. The proof is complete. □

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